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A CONSTRUCTIVE ALGORITHM FOR THE NORMALIZATION OF A PERIODIC HAMILTONIAN†

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A time-periodic Hamiltonian system is considered. It is assumed that the system has an equilibrium position in whose neighbourhood the Hamiltonian is analytic. A constructive algorithm is proposed for computing the coefficients of the normal form of the Hamiltonian. The algorithm is based on a special procedure for the construction and analysis of a symplectic map of the neighbourhood of the equilibrium position onto itself. The exposition is carried out using as an example a system with two degrees of freedom. The coefficients of the normal form are expressed in terms of the coefficients of the generating function of the map. The algorithm is used to solve the problem of the stability of the relative equilibrium of a Kovalevskaya top with a vertically oscillating suspension point.

In many stability problems for the motion and non-linear oscillations of mechanical systems, it is necessary to investigate the behaviour of trajectories of a canonical system of differential equations in the neighbourhood of a point of equilibrium which coincides with the origin of the phase space. In such cases the Hamiltonian is frequently periodic in time or not explicitly time-dependent.

One of the main technical devices for such investigation is Poincaré's method of normal forms, which has been extensively developed and used in a large variety of non-linear problems [1–3]. The essence of the method is to use a canonical transformation to bring the Hamiltonian to a certain simpler (normal) form. The corresponding canonical system of differential equations is considerable simplified, significantly facilitating its investigation.

If the Hamiltonian is not explicitly time-dependent, its normal form may be obtained by algebraic operations applied to the coefficients of the series expansion of the Hamiltonian in the neighbourhood of the equilibrium point [1, 2, 4]. For example, the conditions for the stability and instability of the equilibrium position may be expressed explicitly in terms of the coefficients of the initial Hamiltonian [4].

However, if the Hamiltonian is explicitly time-dependent, the derivation of the normal form involves a rather complicated procedure. The first stage involves the construction of a time-periodic linear canonical transformation to normalize the part of the Hamiltonian that is quadratic in the phase variables. Then the terms of the third and higher powers in the series expansion of the Hamiltonian must be normalized. The non-linear canonical transformation is close to the identity and is defined by series with time-periodic coefficients, which are constructed using the Birkhoff transformation [5] or its modern modifications, such as the Deprit–Hori transformation [6]. The construction of these series is extremely laborious. The technical aspect of the normalization procedure may be simplified considerably by using the method of point mappings (see [4, Chap. 6]).

In the algorithm proposed here, as in an earlier version [4], what is normalized is not the time-periodic Hamiltonian itself, but the generating function of a certain map, generated by the canonical system of differential equations corresponding to the Hamiltonian, over a period. It is then possible to reproduce the normal form of the Hamiltonian on the basis of the normal form of the generating function.

As before [4], the construction of the map is based on solving a Hamilton–Jacobi equation in the neighbourhood of the equilibrium point in series form. However, unlike the algorithm in [4], there is no need for preliminary normalization of the quadratic part of the original Hamiltonian.

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The algorithm is very simple – not much more complicated than the algorithm for the normalization of an autonomous Hamiltonian system. True, the algorithm must, as a rule, be run using computers. However, the coefficients of the series expansion of the generating function of the map are obtained by integrating a system of ordinary differential equations only once over the period; that system is very easy to derive from the initial Hamiltonian, while the initial conditions are known in advance. As regards the coefficients of the normal form of the Hamiltonian, they are explicitly expressed in terms of the coefficients of the series expansion of the generating function of the map.

1. THE ALGORITHM FOR THE NORMALIZATION OF A PERIODIC HAMILTONIAN

Construction of the map. Consider a system with two degrees of freedom whose motion is described by canonical equations with a Hamiltonian $H(q_1, q_2, p_1, p_2, t)$. We shall assume that H is analytic in the neighbourhood of the point $q_j = p_j = 0$ $(j = 1, 2)$, which corresponds to an equilibrium point of the system, and that it admits of a series expansion

$$H = H_2 + H_3 + H_4 + \dots \tag{1.1}$$

where H_k is a form of degrees k in q_1, q_2, p_1 and p_2 whose coefficients are 2π-periodic functions of t.

Let $q_j^{(0)}$ and $p_j^{(0)}$ $(j = 1, 2)$ be the initial values of the variables q_j and p_j, and $q_j^{(1)}$ and $p_j^{(1)}$ are their values at $t = 2\pi$. If $q_j^{(0)}$ and $p_j^{(0)}$ are sufficiently small, the quantities $q_j^{(1)}$ and $p_j^{(1)}$ will be analytic functions of $q_1^{(0)}, q_2^{(0)}, p_1^{(0)}$ and $p_2^{(0)}$, defining a map T of the neighbourhood of the equilibrium position onto itself. We will now outline an algorithm for constructing this map.

Let $\mathbf{X}(t)$ be the fundamental matrix of solutions of the linearized equations of motion. Its elements satisfy the equations

$$\frac{dx_{js}}{dt} = \frac{\partial H_2}{\partial x_{j+2,s}}, \quad \frac{dx_{j+2,s}}{dt} = -\frac{\partial H_2}{\partial x_{js}}, \quad H_2 = H_2(x_{1s}, x_{2s}, x_{3s}, x_{4s}, t);$$
$$j = 1, 2; \quad s = 1, 2, 3, 4 \tag{1.2}$$

and the initial conditions

$$\mathbf{X}(0) = \mathbf{E}_4 \tag{1.3}$$

where $\mathbf{E}_4$ is the 4×4 identity matrix.

Instead of the variables q_j and p_j $(j = 1, 2)$, we will introduce new canonical conjugate variables u_j and υ_j by the formula

$$\begin{Vmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{Vmatrix} = \mathbf{X}(t) \begin{Vmatrix} u_1 \\ u_2 \\ \upsilon_1 \\ \upsilon_2 \end{Vmatrix} \tag{1.4}$$

This change of variables is a canonical univalent transformation [7]. The series expansion of the new Hamiltonian $G(u_1, u_2, \upsilon_1, \upsilon_2, t)$ contains no quadratic terms in u_1, u_2, υ_1 and υ_2:

$$G = G_3 + G_4 + \dots \tag{1.5}$$

where G_k is the forma H_k of (1.1) in which the highest-order variables expressed in terms of the new ones by formula (1.4).

The change of variables (1.4) reduces the construction of the map T to finding the map $q_j^{(0)}, p_j^{(0)} \to u_j^{(1)}, \upsilon_j^{(1)}$ over a period, i.e. for t varying from 0 to 2π. In this situation we have $q_j^{(0)} = u_j^{(0)}, p_j^{(0)} = \upsilon_j^{(0)}$, and

$$\begin{Vmatrix} q_1^{(1)} \\ q_2^{(1)} \\ p_1^{(1)} \\ p_2^{(1)} \end{Vmatrix} = \mathbf{X}(2\pi) \begin{Vmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \upsilon_1^{(1)} \\ \upsilon_2^{(1)} \end{Vmatrix} \tag{1.6}$$

Since expansion (1.5) contains no second-order terms, the map $q_j^{(0)}, p_j^{(0)} \to u_j^{(1)}, \upsilon_j^{(1)}$ is close to an identity. It is defined implicitly by the equalities

$$\begin{gathered} q_j^{(0)} = \frac{\partial S}{\partial p_j^{(0)}}, \quad \upsilon_j^{(1)} = \frac{\partial S}{\partial u_j^{(1)}}; \quad j = 1, 2 \\ S = u_1^{(1)} p_1^{(0)} + u_2^{(1)} p_2^{(0)} + S_3(u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}) + S_4(u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}) + \dots \end{gathered} \tag{1.7}$$

where S is the value at $t = 2\pi$ of the function

$$\Phi = u_1^{(1)} p_1^{(0)} + u_2^{(1)} p_2^{(0)} + \Phi_3(u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}, t) + \Phi_4(u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}, t) + \dots \tag{1.8}$$

which satisfies the Hamilton–Jacobi equation

$$\frac{\partial \Phi}{\partial t} + G\left(u_1^{(1)}, u_2^{(1)}, \frac{\partial \Phi}{\partial u_1^{(1)}}, \frac{\partial \Phi}{\partial u_2^{(1)}}, t\right) = 0; \quad \Phi_k(u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}, 0) \equiv 0, \quad k = 3, 4, \dots \tag{1.9}$$

Substituting expansions (1.5) and (1.8) into the left-hand side of Eq. (1.9) and equating terms of powers 3, 4, etc. to zero, we obtain equations for the forms $\Phi_3, \Phi_4, \dots$:

$$\frac{\partial \Phi_3}{\partial t} = -G_3, \quad \frac{\partial \Phi_4}{\partial t} = -G_4 - \sum_{j=1}^{2} \frac{\partial G_3}{\partial p_j^{(0)}} \frac{\partial \Phi_3}{\partial u_j^{(1)}}, \dots; \quad G_k = G_k(u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}, t) \tag{1.10}$$

Equating the coefficients of like powers of $u_1^{(1)}, u_2^{(1)}, p_1^{(0)}$ and $p_2^{(0)}$ on the left and right of these equations, we obtain a system of ordinary differential equations for the coefficients of the forms $\Phi_3, \Phi_4, \dots$. By the identities of (1.9), these coefficients vanish at $t = 0$. The equations for the coefficients must be considered together with the system of equations (1.2), (1.3), defining the elements of the fundamental matrix $\mathbf{X}(t)$, which occur in the substitution (1.4) and therefore also in the expressions for the functions $G_3, G_4, \dots$. Integration of the system thus obtained from $t = 0$ to $t = 2\pi$ yields the functions $S_3, S_4, \dots$, and hence also, (1.6) and (1.7), the explicit form of the map T:

$$\begin{Vmatrix} q_1^{(1)} \\ q_2^{(1)} \\ p_1^{(1)} \\ p_2^{(1)} \end{Vmatrix} = \mathbf{X}(2\pi) \begin{Vmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{p}_1 \\ \tilde{p}_2 \end{Vmatrix}$$

$$\begin{gathered} \tilde{q}_j = q_j^{(0)} - \frac{\partial S_3}{\partial p_j^{(0)}} + \sum_{l=1}^{2} \frac{\partial^2 S_3}{\partial p_j^{(0)} \partial q_l^{(0)}} \frac{\partial S_3}{\partial p_l^{(0)}} - \frac{\partial S_4}{\partial p_j^{(0)}} + O_4 \\ \tilde{p}_j = p_j^{(0)} + \frac{\partial S_3}{\partial q_j^{(0)}} - \sum_{l=1}^{2} \frac{\partial^2 S_3}{\partial q_j^{(0)} \partial q_l^{(0)}} \frac{\partial S_3}{\partial p_l^{(0)}} + \frac{\partial S_4}{\partial q_j^{(0)}} + O_4 \end{gathered} \tag{1.11}$$

$$S_k = S_k(q_1^{(0)}, q_2^{(0)}, p_1^{(0)}, p_2^{(0)}); \quad j = 1, 2; \quad k = 3, 4$$

where O_4 denotes terms of degree greater than 3 in $q_1^{(0)}, q_2^{(0)}, p_1^{(0)}$ and $p_2^{(0)}$.

Linear normalization of the map (1.11). The characteristic equation of the matrix $\mathbf{X}(2\pi)$ of the linearized map (1.11) is reciprocal and has the form

$$\varrho^4 - a_1\varrho^3 + a_2\varrho^2 - a_1\varrho + 1 = 0 \tag{1.12}$$

where a_1 is the trace of the matrix $\mathbf{X}(2\pi)$ and a_2 is the sum of all its principal minors of the second order. We shall consider only the case in which the parameters of the system lie in the interior of the stable domain of the equilibrium position $q_j = p_j = 0$ ($j = 1, 2$) in the first approximation. In the plane of the coefficients a_1 and a_2 this domain is defined by the following system of inequalities [8]

$$-2 < a_2 < 6, \quad 4(a_2 - 2) < a_1^2 < (a_2 + 2)^2/4 \tag{1.13}$$

When these inequalities hold, the roots of Eq. (1.12) are complex conjugates, distinct and of absolute value 1. The characteristic indices $\pm i\lambda_j$ ($j = 1, 2$) will be pure imaginary.

In this section we will use a change of variables to bring the linear part of the map (1.11) to real normal form. This transformation may be constructed as follows. Assign (arbitrary) signs to the quantities λ_j ($j = 1, 2$) and let $\mathbf{e}_j$ denote an eigenvector of the matrix $\mathbf{X}(2\pi)$ corresponding to the root (multiplier) $\varrho_j = e^{i2\pi\lambda_j}$ of Eq. (1.12). For the real and imaginary parts $\mathbf{r}_j$ and $\mathbf{s}_j$ of the vector $\mathbf{e}_j$ we have the following system of equations

$$\mathbf{X}(2\pi)\mathbf{r}_j = \cos 2\pi\lambda_j\mathbf{r}_j - \sin 2\pi\lambda_j\mathbf{s}_j, \quad \mathbf{X}(2\pi)\mathbf{s}_j = \cos 2\pi\lambda_j\mathbf{s}_j + \sin 2\pi\lambda_j\mathbf{r}_j \tag{1.14}$$

Let $\mathbf{r}_j^*$, $\mathbf{s}_j^*$ be some non-trivial solution of system (1.14). Let g_j denote the scalar product of the vectors $\mathbf{r}_j^*$ and $\mathbf{I}\mathbf{s}_j^*$, that is,

$$g_j = (\mathbf{r}_j^*, \mathbf{I}\mathbf{s}_j^*), \quad \mathbf{I} = \begin{Vmatrix} \mathbf{0} & \mathbf{E}_2 \\ -\mathbf{E}_2 & \mathbf{0} \end{Vmatrix}$$

where $\mathbf{E}_2$ is the 2×2 identity matrix. It can be shown [4] that the quantities g_j ($j = 1, 2$) do not vanish.

We introduce the notation

$$\delta_j = \operatorname{sign} g_j, \quad \sigma_j = \delta_j\lambda_j, \quad c_j = |g_j|^{-1/2}, \quad j = 1, 2 \tag{1.15}$$

and we let $\mathbf{N}$ denote the 4×4 matrix whose jth and $(j + 2)$th columns are $c_j\delta_j\mathbf{r}_j^*$ and $c_j\mathbf{s}_j^*$ ($j = 1, 2$), respectively.

It can be verified directly that the matrix $\mathbf{N}$ is symplectic and transforms the matrix $\mathbf{X}(2\pi)$ to real normal form $\mathbf{G}$:

$$\mathbf{N}'\mathbf{I}\mathbf{N} = \mathbf{I}, \quad \mathbf{N}^{-1}\mathbf{X}(2\pi)\mathbf{N} = \mathbf{G}$$

$$\mathbf{G} = \begin{Vmatrix} \mathbf{G}_c & \mathbf{G}_s \\ -\mathbf{G}_s & \mathbf{G}_c \end{Vmatrix}, \quad \mathbf{G}_c = \begin{Vmatrix} \cos 2\pi\sigma_1 & 0 \\ 0 & \cos 2\pi\sigma_2 \end{Vmatrix}, \quad \mathbf{G}_s = \begin{Vmatrix} \sin 2\pi\sigma_1 & 0 \\ 0 & \sin 2\pi\sigma_2 \end{Vmatrix}$$

The matrix $\mathbf{G}$ defines two independent rotations through angles $2\pi\sigma_1$ and $2\pi\sigma_2$.

Instead of q_j and p_j ($j = 1, 2$) in the map (1.11) we define new variables Q_j and P_j ($j = 1, 2$) via the univalent canonical transformation defined by the matrix $\mathbf{N}$

$$\begin{Vmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{Vmatrix} = \mathbf{N} \begin{Vmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{Vmatrix} \tag{1.16}$$

Omitting the intermediate steps, we write the expression for the map (1.11) in the new variables

$$\left\|\begin{matrix} Q_1^{(1)} \\ Q_2^{(1)} \\ P_1^{(1)} \\ P_2^{(1)} \end{matrix}\right\| = \mathbf{G}\left\|\begin{matrix} \tilde{Q}_1 \\ \tilde{Q}_2 \\ \tilde{P}_1 \\ \tilde{P}_2 \end{matrix}\right\| \tag{1.17}$$

$$\tilde{Q}_j = Q_j^{(0)} - \frac{\partial F_3}{\partial P_j^{(0)}} + \sum_{l=1}^{2} \frac{\partial^2 F_3}{\partial P_j^{(0)} \partial Q_l^{(0)}} \frac{\partial F_3}{\partial P_l^{(0)}} - \frac{\partial F_4}{\partial P_j^{(0)}} + O_4$$

$$\tilde{P}_j = P_j^{(0)} + \frac{\partial F_3}{\partial Q_j^{(0)}} - \sum_{l=1}^{2} \frac{\partial^2 F_3}{\partial Q_j^{(0)} \partial Q_l^{(0)}} \frac{\partial F_3}{\partial P_l^{(0)}} + \frac{\partial F_4}{\partial Q_j^{(0)}} + O_4; \quad j = 1, 2$$

We have used the notation

$$F_3 = S_3^* \tag{1.18}$$

$$F_4 = S_4^* + \frac{1}{2}\sum_{j=1}^{2}\left[(n_{1,2+j}n_{3,2+j} + n_{2,2+j}n_{4,2+j})\left(\frac{\partial S_3^*}{\partial Q_j^{(0)}}\right)^2 + (n_{1j}n_{3j} + n_{2j}n_{4j})\left(\frac{\partial S_3^*}{\partial P_j^{(0)}}\right)^2\right] -$$

$$- \sum_{j=1}^{2}\left[(n_{13}n_{3j} + n_{23}n_{4j})\frac{\partial S_3^*}{\partial Q_1^{(0)}}\frac{\partial S_3^*}{\partial P_j^{(0)}} + (n_{14}n_{3j} + n_{24}n_{4j})\frac{\partial S_3^*}{\partial Q_2^{(0)}}\frac{\partial S_3^*}{\partial P_j^{(0)}}\right] +$$

$$+ (n_{13}n_{34} + n_{23}n_{44})\frac{\partial S_3^*}{\partial Q_1^{(0)}}\frac{\partial S_3^*}{\partial Q_2^{(0)}} + (n_{11}n_{32} + n_{21}n_{42})\frac{\partial S_3^*}{\partial P_1^{(0)}}\frac{\partial S_3^*}{\partial P_2^{(0)}} \tag{1.19}$$

where n_{rs} are the elements of the matrix $\mathbf{N}$ and S_k^* ($k = 3, 4$) are the forms S_k from (1.11), with $q_j^{(0)}$ and $p_j^{(0)}$ expressed in terms of $Q_j^{(0)}$ and $P_j^{(0)}$ in accordance with the transformation (1.16).

Corresponding to the linearized map (1.17) we have the normal form H_2^* of the quadratic part H_2 of the initial Hamiltonian (1.1)

$$H_2^* = \frac{1}{2}\sigma_1(Q_1^2 + P_1^2) + \frac{1}{2}\sigma_2(Q_2^2 + P_2^2) \tag{1.20}$$

Non-linear normalization of the map. Non-linear normalization is more conveniently done in complex variables. We apply a univalent canonical transformation $Q_1, Q_2, P_1, P_2 \to x_1, x_2, y_1, y_2$ to (1.17), where

$$Q_j = \frac{1+i}{2}(x_j + y_j), \quad P_j = -\frac{1-i}{2}(x_j - y_j); \quad j = 1, 2 \tag{1.21}$$

where i is the square root of –1.

In complex variables x_j, y_j the map (1.17) becomes

$$\begin{aligned} x_j^{(1)} &= \varrho_j\left(x_j^{(0)} - \frac{\partial Z_3}{\partial y_j^{(0)}} + \sum_{l=1}^{2} \frac{\partial^2 Z_3}{\partial y_j^{(0)} \partial x_l^{(0)}} \frac{\partial Z_3}{\partial y_l^{(0)}} - \frac{\partial Z_4}{\partial y_j^{(0)}} + O_4\right) \\ y_j^{(1)} &= \varrho_{j+2}\left(y_j^{(0)} + \frac{\partial Z_3}{\partial x_j^{(0)}} - \sum_{l=1}^{2} \frac{\partial^2 Z_3}{\partial x_j^{(0)} \partial x_l^{(0)}} \frac{\partial Z_3}{\partial y_l^{(0)}} + \frac{\partial Z_4}{\partial x_j^{(0)}} + O_4\right); \quad j = 1, 2 \end{aligned} \tag{1.22}$$

where

$$\varrho_j = e^{i2\pi\sigma_j}, \quad \varrho_{j+2} = e^{-i2\pi\sigma_j}; \quad j = 1, 2 \tag{1.23}$$

are the roots of the characteristic equation (1.12), and

$$Z_3 = F_3^*, \quad Z_4 = F_4^* + \frac{1}{4}\sum_{j=1}^{2}\left[\left(\frac{\partial F_3^*}{\partial x_j^{(0)}}\right)^2 - \left(\frac{\partial F_3^*}{\partial y_j^{(0)}}\right)^2 + 2\frac{\partial F_3^*}{\partial x_j^{(0)}}\frac{\partial F_3^*}{\partial y_j^{(0)}}\right] \tag{1.24}$$

where F_k^* $(k = 1, 2)$ are the forms F_k defined by (1.18) and (1.19) in which $Q_j^{(0)}$ and $P_j^{(0)}$ have been expressed in terms of $x_j^{(0)}$ and $y_j^{(0)}$ by formulae (1.21). The forms Z_k in relations (1.24) will be written as sums

$$Z_k = \sum z_{\nu_1\nu_2\mu_1\mu_2} x_1^{(0)^{\nu_1}} x_2^{(0)^{\nu_2}} y_1^{(0)^{\mu_1}} y_2^{(0)^{\mu_2}}; \quad k = 3, 4$$

where the summation is carried out over non-negative integers υ_1, υ_2, μ_1 and μ_2 that add up to k (and similarly in what follows, when analogous representations are used for forms).

Normalization of the map (1.22) in second-degree terms. We replace the variables x_j, y_j $(j = 1, 2)$ by new variables ξ_j, η_j $(j = 1, 2)$, using the generating function $R(x_1, x_2, \eta_1, \eta_2)$ defined by

$$R = x_1\eta_1 + x_2\eta_2 + R_3 + R_4 + \ldots; \quad R_s = \sum r_{\nu_1\nu_2\mu_1\mu_2} x_1^{\nu_1} x_2^{\nu_2} \eta_1^{\mu_1} \eta_2^{\mu_2} \tag{1.25}$$

The equalities

$$y_j = \frac{\partial R}{\partial x_j}, \quad \xi_j = \frac{\partial R}{\partial \eta_j}; \quad j = 1, 2$$

yield explicit expressions for the old variables in terms of the new ones

$$\begin{aligned} x_j &= \xi_j - \frac{\partial R_3}{\partial \eta_j} + \sum_{l=1}^{2}\frac{\partial^2 R_3}{\partial \eta_j \partial \xi_l}\frac{\partial R_3}{\partial \eta_l} - \frac{\partial R_4}{\partial \eta_j} + O_4 \\ y_j &= \eta_j + \frac{\partial R_3}{\partial \xi_j} - \sum_{l=1}^{2}\frac{\partial^2 R_3}{\partial \xi_j \partial \xi_l}\frac{\partial R_3}{\partial \eta_l} + \frac{\partial R_4}{\partial \xi_j} + O_4; \quad j = 1, 2 \end{aligned} \tag{1.26}$$

where R_k are the functions in (1.25), with the variables x_j replaced by ξ_j.

Using equalities (1.26), we express $x_j^{(1)}, y_j^{(1)}$ and $x_j^{(0)}, y_j^{(0)}$ in terms of $\xi_j^{(1)}$, $\eta_j^{(1)}$ and $\xi_j^{(0)}$, $\eta_j^{(0)}$, respectively, and substitute them into relations (1.22). Solving the equations thus obtained for $\xi_j^{(1)}$ and $\eta_j^{(1)}$, we obtain the map in the new variables

$$\xi_j^{(1)} = \varrho_j\left(\xi_j^{(0)} - \frac{\partial W_3}{\partial \eta_j^{(0)}} + \ldots\right), \quad \eta_j^{(1)} = \varrho_{j+2}\left(\eta_j^{(0)} + \frac{\partial W_3}{\partial \xi_j^{(0)}} + \ldots\right); \quad j = 1, 2 \tag{1.27}$$

where the dots stand for the terms of power greater than two in $\xi_1^{(0)}$, $\xi_2^{(0)}$, $\eta_1^{(0)}$ and $\eta_2^{(0)}$, and

$$\begin{aligned} W_3 = {} & Z_3(\xi_1^{(0)}, \xi_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)}) + R_3(\xi_1^{(0)}, \xi_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)}) - \\ & - R_3(\varrho_1\xi_1^{(0)}, \varrho_2\xi_2^{(0)}, \varrho_3\eta_1^{(0)}, \varrho_4\eta_2^{(0)}) \end{aligned} \tag{1.28}$$

The function R_3 is chosen in such a way as to simplify (or even to eliminate) the second degree terms in (1.27) as far as possible.

We write W_3 as a sum

$$W_3 = \sum w_{\nu_1\nu_2\mu_1\mu_2}\xi_1^{(0)^{\nu_1}}\xi_2^{(0)^{\nu_2}}\eta_1^{(0)^{\mu_1}}\eta_2^{(0)^{\mu_2}}$$

Equations (1.23) and (1.28) imply the following expressions for the coefficients

$$w_{\nu_1\nu_2\mu_1\mu_2} = z_{\nu_1\nu_2\mu_1\mu_2} + \left(1 - e^{i2\pi l_{\nu_1\nu_2\mu_1\mu_2}}\right) r_{\nu_1\nu_2\mu_1\mu_2}; \quad l_{\nu_1\nu_2\mu_1\mu_2} = (\nu_1 - \mu_1)\sigma_1 + (\nu_2 - \mu_2)\sigma_2 \tag{1.29}$$

In the interior of the stable domain (1.13) of the linearized map, there can be no resonances of up to and including two. Let us assume that there are also no third-order resonances, i.e. that there can be no equality

$$k_1\sigma_1 + k_2\sigma_2 = n \tag{1.30}$$

where n is an arbitrary integer, and k_1 and k_2 are integers such that $|k_1| + |k_2| = 3$. Then the number $l_{\nu_1\nu_2\mu_1\mu_2}$ in (1.29) will not be an integer and, putting

$$r_{\nu_1\nu_2\mu_1\mu_2} = \frac{z_{\nu_1\nu_2\mu_1\mu_2}}{e^{i2\pi l_{\nu_1\nu_2\mu_1\mu_2}} - 1} \tag{1.31}$$

we get $w_{\nu_1\nu_2\mu_1\mu_2} = 0$. Then $W_3 = 0$, and there will be no second degree terms in the normalized map (1.27).

Now suppose that there is one third-order resonance in the system. We shall consider not arbitrary resonances, but only resonances for which the numbers k_1 and k_2 in (1.30) satisfy the inequality $k_1k_2 \geq 0$. Only such resonances may cause a system that is stable in the first approximation to become unstable in the non-linear approximation [9]. Thus, we shall assume that one of the following four resonance relations holds in the system

$$1)\ 3\sigma_1 = n, \quad 2)\ 3\sigma_2 = n, \quad 3)\ \sigma_1 + 2\sigma_2 = n, \quad 4)\ 2\sigma_1 + \sigma_2 = n \tag{1.32}$$

Then the two monomials in W_3 for which $l_{\nu_1\nu_2\mu_1\mu_2}$ equals n or $-n$ cannot be made to vanish. The map normalized in second-degree terms will be defined by equalities (1.26) in which

$$W_3 = z_{k_1k_200}\xi_1^{(0)^{k_1}}\xi_2^{(0)^{k_2}} + z_{00k_1k_2}\eta_1^{(0)^{k_1}}\eta_2^{(0)^{k_2}} \tag{1.33}$$

Normalization of the map in third-degree terms. Suppose that there are no third-order resonances. Choosing the coefficients of the form R_3 according to formula (1.31), we eliminate all second-degree terms in the map (1.27). Calculations show that with this choice of R_3 the map may be written as

$$\xi_j^{(1)} = \varrho_j\left(\xi_j^{(0)} - \frac{\partial W_4}{\partial\eta_j^{(0)}} + O_4\right), \quad \eta_j^{(1)} = \varrho_{j+2}\left(\eta_j^{(0)} + \frac{\partial W_4}{\partial\xi_j^{(0)}} + O_4\right); \quad j = 1, 2 \tag{1.34}$$

$$W_4 = Z_4(\xi_1^{(0)}, \xi_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)}) + \sum_{j=1}^{2} \frac{\partial Z_3(\xi_1^{(0)}, \xi_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)})}{\partial\eta_j^{(0)}} \frac{\partial R_3(\xi_1^{(0)}, \xi_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)})}{\partial\xi_j^{(0)}} +$$
$$+ R_4(\xi_1^{(0)}, \xi_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)}) - R_4(\varrho_1\xi_1^{(0)}, \varrho_2\xi_2^{(0)}, \varrho_3\eta_1^{(0)}, \varrho_4\eta_2^{(0)}) \tag{1.35}$$

Suppose there are no fourth-order resonances in the system. One might try to choose the form R_4 so as to eliminate the third-degree terms in the map (1.34). However, this cannot be done. As is obvious from expressions (1.29), the terms in W_4 for which $\nu_1 = \mu_1$, $\nu_2 = \mu_2$ cannot be eliminated. The map normalized in third-degree terms may be written as equalities (1.34) in which

$$W_4 = w_{2020}\xi_1^{(0)^2}\eta_1^{(0)^2} + w_{1111}\xi_1^{(0)}\xi_2^{(0)}\eta_1^{(0)}\eta_2^{(0)} + w_{0202}\xi_2^{(0)^2}\eta_2^{(0)^2} \tag{1.36}$$

The coefficients of the form (1.36) are real numbers. They are expressed in terms of the coefficients of the forms F_3 and F_4 from (1.18) and (1.19) by formulae (3.7)–(3.9) of Section 3.

Now suppose that there is a fourth-order resonance in the system, that is, equality (1.30) holds with $|k_1| + |k_2| = 4$. As in the case of third-order resonance, we will confine our attention to single resonances, and only to those for which the numbers k_1 and k_2 are non-negative. The following five such resonances are possible

$$1)\ 4\sigma_1 = n, \quad 2)\ 4\sigma_2 = n, \quad 3)\ 2(\sigma_1+\sigma_2) = n, \quad 4)\ \sigma_1 + 3\sigma_2 = n, \quad 5)\ 3\sigma_1 + \sigma_2 = n \tag{1.37}$$

For each of these resonances, the form W_4 in the normalized map (1.34) will contain, apart from non-vanishing monomials of the form (1.36), also two monomials characteristic for that specific resonance

$$\begin{aligned} W_4 = {} & w_{2020}\xi_1^{(0)^2}\eta_1^{(0)^2} + w_{1111}\xi_1^{(0)}\xi_2^{(0)}\eta_1^{(0)}\eta_2^{(0)} + w_{0202}\xi_2^{(0)^2}\eta_2^{(0)^2} + \\ & + w_{k_1k_200}\xi_1^{(0)^{k_1}}\xi_2^{(0)^{k_2}} + w_{00k_1k_2}\eta_1^{(0)^{k_1}}\eta_2^{(0)^{k_2}} \end{aligned} \tag{1.38}$$

The last (resonant) coefficients in the form (1.38) are complex conjugates

$$w_{k_1k_200} = \mu_{k_1k_200} - i\nu_{k_1k_200}, \quad w_{00k_1k_2} = \mu_{k_1k_200} + i\nu_{k_1k_200} \tag{1.39}$$

Expressions for the quantities $\mu_{k_1k_200}$ and $\nu_{k_1k_200}$ are given in Section 3 (formulae (3.10)–(3.19)).

The normal form of the Hamiltonian. Given the normal form of the map, it is now quite easy to construct a 2π-periodic function of t, $\Gamma(\xi_1, \xi_2, \eta_1, \eta_2, t)$, which is the normal form of the original Hamiltonian (1.1). If there are no resonances of order up to and including four, then

$$\Gamma = i\sigma_1\xi_1\eta_1 + i\sigma_2\xi_2\eta_2 - \frac{1}{2\pi}(w_{2020}\xi_1^2\eta_1^2 + w_{1111}\xi_1\xi_2\eta_1\eta_2 + w_{0202}\xi_2^2\eta_2^2) + O_5 \tag{1.40}$$

where O_5 are the terms of degree greater than four in ξ_j, η_j, and w_{0202}, w_{1111} and w_{0202} are coefficients of the form (1.36).

If there is a single third-order resonance (see Eqs (1.30) and (1.32)), then

$$\Gamma = i\sigma_1\xi_1\eta_1 + i\sigma_2\xi_2\eta_2 - \frac{1}{2\pi}(z_{k_1k_200}e^{-int}\xi_1^{k_1}\xi_2^{k_2} + z_{00k_1k_2}e^{int}\eta_1^{k_1}\eta_2^{k_2}) + O_4 \tag{1.41}$$

where $z_{k_1k_200}$ and $z_{00k_1k_2}$ are coefficients of the form (1.33).

If there are no third-order resonances but there is a single fourth-order resonance (see Eqs (1.30) and (1.37)), then

$$\begin{aligned} \Gamma = {} & i\sigma_1\xi_1\eta_1 + i\sigma_2\xi_2\eta_2 - \frac{1}{2\pi}(w_{2020}\xi_1^2\eta_1^2 + w_{1111}\xi_1\xi_2\eta_1\eta_2 + w_{0202}\xi_2^2\eta_2^2 + \\ & + w_{k_1k_200}e^{-int}\xi_1^{k_1}\xi_2^{k_2} + w_{00k_1k_2}e^{int}\eta_1^{k_1}\eta_2^{k_2}) + O_5 \end{aligned} \tag{1.42}$$

where $w_{\nu_1\nu_2\mu_1\mu_2}$ are coefficients of the form (1.38).

In real canonically conjugate variables r_j, φ_j ($j = 1, 2$), defined by a univalent canonical transformation

$$\xi_j = -\frac{1+i}{2}\sqrt{2r_j}e^{i\varphi_j}, \quad \eta_j = \frac{1+i}{2}\sqrt{2r_j}e^{-i\varphi_j}; \quad i = 1, 2 \tag{1.43}$$

the normalized Hamiltonians (1.30), (1.41), (1.42) become, respectively

$$H = \sigma_1 r_1 + \sigma_2 r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + O((r_1+r_2)^{5/2}) \tag{1.44}$$

$$H = \sigma_1 r_1 + \sigma_2 r_2 + \frac{\sqrt{2}}{4\pi}r_1^{k_1/2}r_2^{k_2/2}(\alpha_{k_1k_200}\sin\gamma + \beta_{k_1k_200}\cos\gamma) + O((r_1+r_2)^2) \tag{1.45}$$

$$H = \sigma_1 r_1 + \sigma_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + r_1^{k_1/2} r_2^{k_2/2} (\alpha_{k_1 k_2 00} \sin\gamma + \\ + \beta_{k_1 k_2 00} \cos\gamma) + O((r_1 + r_2)^{5/2}) \quad (1.46)$$

where

$$c_{20} = \frac{1}{2\pi} w_{2020}, \quad c_{11} = \frac{1}{2\pi} w_{1111}, \quad c_{02} = \frac{1}{2\pi} w_{0202} \quad (1.47)$$

$$\alpha_{k_1 k_2 00} = \frac{1}{\pi} \nu_{k_1 k_2 00}, \quad \beta_{k_1 k_2 00} = \frac{1}{\pi} \mu_{k_1 k_2 00}, \quad \gamma = k_1 \varphi_1 + k_2 \varphi_2 - nt \quad (1.48)$$

2. THE STABILITY OF THE RELATIVE EQUILIBRIUM OF A RIGID BODY UNDER OSCILLATIONS OF ITS SUSPENSION POINT

Consider a rigid body moving in a uniform field of gravity. Let $O_* X_* Y_* Z_*$ be a fixed system of coordinates whose $O_* Z_*$ axis points vertically upwards. Suppose one point O of the body is moving along the vertical $O_* Z_*$ according to a harmonic law $O_* O = -a\cos(\Omega t)$ $(a > 0)$. Let mg be the weight of the body and let $\mathbf{l}$ be the radius vector of the centre of gravity relative to the point O. Let $Oxyz$ be a system of coordinates moving with the body, its axes directed along the principal axes of inertia of the body for the point O. The moments of inertia are A, B and C. One further system of coordinates $OXYZ$ is moving linearly with its axes parallel to the corresponding axes of the system $O_* X_* Y_* Z_*$.

When the body's centre of gravity lies on the vertical $O_* Z_*$, it has two relative equilibrium positions (in the system $OXYZ$). One corresponds to the normal position of the body (with the centre of gravity below the point O), and the other to the inverted position (with the centre of gravity above O). We will investigate the problem of the stability of these equilibrium positions of the body. Let us assume that the body has the mass geometry of a Kovalevskaya top. Then $A = B = 2C$, and the centre of gravity may be assumed to lie on the Ox axis.

The Hamiltonian. The mutual orientation of the trihedrals $Oxyz$ and $OXYZ$ will be defined in terms of the Euler angles ψ, θ, φ.. Let $\mathbf{v}_0$ be the velocity of the point O of the body, and let p, q and r be the components of the angular velocity vector of the body in the system of coordinates $Oxyz$. The kinetic and potential energy are given by the formulae

$$T = \frac{1}{2} m v_0^2 + m(\mathbf{v}_0, \boldsymbol{\omega} \times \mathbf{l}) + \frac{1}{2} C(2p^2 + 2q^2 + r^2), \quad \Pi = mgl \sin\theta \sin\varphi$$

Dropping terms independent of ψ, θ, φ and their derivatives with respect to time, we obtain the following expression for the Lagrangian $L = T - \Pi$

$$L = C(\dot\psi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} C(\dot\psi \cos\theta + \dot\varphi)^2 + ma\Omega l \sin(\Omega t)(\dot\varphi \sin\theta \cos\varphi + \\ + \dot\theta \cos\theta \sin\varphi) - mgl \sin\theta \sin\varphi \quad (2.1)$$

The generalized momenta are evaluated in the usual way

$$p_\psi = \frac{\partial L}{\partial \dot\psi}, \quad p_\theta = \frac{\partial L}{\partial \dot\theta}, \quad p_\varphi = \frac{\partial L}{\partial \dot\varphi} \quad (2.2)$$

The coordinate ψ is cyclic, and therefore p_ψ = const; we shall assume that $p_\psi = 0$. Then, using Eqs (2.1) and (2.2), we obtain the Hamiltonian $H = H(\theta, \varphi, p_\theta, p_\varphi, t)$ in the standard way. Introducing a dimensionless "time" variable $\tau = \Omega t$, we transform to new coordinates and momenta q_j and p_j $(j = 1, 2)$ by applying the canonical transformation (of valence $(C\Omega)^{-1}$)

$$\varphi = \frac{3}{2}\pi + q_1, \quad \theta = \frac{\pi}{2} + q_2 \\ p_\varphi = C\Omega p_1 + ma\Omega l \sin\tau \sin q_1, \quad p_\theta = C\Omega p_2 + ma\Omega l \sin\tau \sin q_2 \quad (2.3)$$

In the new variables, the Hamiltonian will be

$$\begin{aligned} H &= \frac{1}{4}\left(p_1 + 2\beta\sin\tau\sin q_1\sin^2\frac{q_2}{2}\right)^2(\mathrm{tg}^2 q_2 + 2) + \\ &+ \frac{1}{4}\left(p_2 + 2\beta\sin\tau\sin q_2\sin^2\frac{q_1}{2}\right)^2 - \alpha\cos q_1\cos q_2 - \beta\cos\tau(\cos q_1 + \cos q_2) \end{aligned} \tag{2.4}$$

where we have introduced the dimensionless parameters

$$\alpha = \frac{mgl}{C\Omega^2}, \quad \beta = \frac{mal}{C}$$

The Hamiltonian of the perturbed motion. The equations of motion with Hamiltonian (2.4) admit of two particular solutions: $q_1 = q_2 = p_1 = p_2 = 0$ and $q_1 = \pi$, $q_2 = p_1 = p_2 = 0$, corresponding to the normal and inverted positions of relative equilibrium. The Hamiltonian of the perturbed motion for the normal equilibrium position is the Hamiltonian (2.4) itself. Its expansion in powers of q_j and p_j has the form (omitting terms independent of q_j and p_j)

$$H = H_2 + H_4 + \ldots \tag{2.5}$$

$$H_2 = \frac{1}{2}p_1^2 + \frac{1}{2}(\alpha + \beta\cos\tau)q_1^2 + \frac{1}{4}p_2^2 + \frac{1}{2}(\alpha + \beta\cos\tau)q_2^2 \tag{2.6}$$

$$H_4 = -\frac{1}{24}(\alpha + \beta\cos\tau)(q_1^4 + q_2^4) + \frac{1}{4}q_2^2(p_1^2 - \alpha q_1^2) + \frac{1}{4}\beta\sin\tau q_1 q_2(q_1 p_2 + 2q_2 p_1) \tag{2.7}$$

For the inverted equilibrium position we introduce perturbations q_j' and p_j' by making the following canonical change of variables

$$q_1 = \pi + q_1', \quad q_2 = q_2', \quad p_1 = p_1', \quad p_2 = p_2' - 2\beta\sin\tau\sin q_2'$$

Replacing τ by $\tau + \pi$ in the corresponding Hamiltonian of perturbed motion, changing the sign of the parameter α and omitting the primes in the notation of the variables q_j' and p_j' we obtain the Hamiltonian (2.4). To analyse the stability of the relative equilibrium positions of the body, therefore, we can take (2.4) as the Hamiltonian of perturbed motion, assuming that $\beta \geq 0$ and α is of arbitrary sign. As a result of the analysis, the half-plane $\beta \geq 0$ is divided into stable and unstable domains. Those of them for which $\alpha \geq 0$, $\beta \geq 0$ will be stable and unstable domains of the normal equilibrium position. The domains for which $\alpha < 0$, $\beta \geq 0$, after mirror reflection in the axis $\alpha = 0$, will define the stable and unstable domains of the inverted equilibrium position.

The results of a stability analysis.

The linear problem. In the first approximation, the equations of perturbed motion for the pairs of canonically conjugate variables q_1, p_1 and q_2, p_2 are separated. The characteristic equation (1.12) takes the form

$$(\varrho^2 - 2A_1\varrho + 1)(\varrho^2 - 2A_2\varrho + 1) = 0$$

$$A_1 = \frac{1}{2}(x_{11}(2\pi) + x_{33}(2\pi)), \quad A_2 = \frac{1}{2}(x_{22}(2\pi) + x_{44}(2\pi))$$

The stable and unstable domains in the plane of the parameters α and β are obtained by applying the two Ince–Strutt diagrams for the Mathieu equation [10].

The stable domains in the first approximation are given by the system of inequalities $|A_1| < 1$, $|A_2| < 1$. If at least one of these inequalities holds with the opposite sign, the system is unstable.

In what follows, in order to avoid dealing with a denumerable set of stable and unstable domains in the half-plane $\beta \geq 0$ of admissible parameter values, we will confine ourselves to the part of the half-plane defined by the inequalities $\alpha \leq 2$, $0 \leq \beta \leq 10$. With these parameter values four stable domains exist in the first approximation. They are the sets of interior points of triangles g_s ($s = 1, \ldots, 4$) whose bases are the segments [0, 1/4], [1/4, 1/2], [1/2, 1], [1, 2] of the axis $\beta = 0$. The vertices Q_s of the triangles opposite the bases are $Q_1(-0.0851, 0.5942)$, $Q_2(0.3687, 0.2547)$, $Q_3(0.9216, 0.9776)$, $Q_4(1.7924, 2.2558)$ (see Fig. 1). The left and right curvilinear boundaries of the triangles g_s are defined by the equations $\alpha = \alpha_s^{(l)}(\beta)$ and $\alpha = \alpha_s^{(r)}(\beta)$, respectively. For small β

$$\alpha_1^{(l)} = -\frac{1}{4}\beta^2 + O(\beta^4), \quad \alpha_1^{(r)} = \frac{1}{4} - \frac{1}{2}\beta + O(\beta^3), \quad \alpha_2^{(l)} = \frac{1}{4} + \frac{1}{2}\beta + O(\beta^3)$$

$$\alpha_2^{(r)} = \frac{1}{2} - \frac{1}{2}\beta + O(\beta^3), \quad \alpha_3^{(l)} = \frac{1}{2} + \frac{1}{2}\beta + O(\beta^3)$$

$$\alpha_3^{(r)} = 1 - \frac{1}{12}\beta^2 + O(\beta^4), \quad \alpha_4^{(l)} = 1 + \frac{5}{12}\beta^2 + O(\beta^4), \quad \alpha_4^{(r)} = 2 - \frac{1}{24}\beta^2 + O(\beta^4)$$

For values of α and β that satisfy the inequalities $\alpha \le 2$, $0 \le \beta \le 10$ and lie outside the domains g_s $(s = 1, \dots, 4)$, one has instability in the strictly non-linear formulation of the problem.

The normal form of the quadratic part (2.6) of the Hamiltonian of perturbed motion has the form (1.20). If $\beta = 0$, we have $\sigma_1 = \sqrt{\alpha}$, $\sigma_2 = \sqrt{\alpha/2}$. Using the continuity of the characteristic exponents, one can derive formulae to compute first approximations of the quantities σ_1 and σ_2 in the stability domains g_s. Putting $c_j = (2\pi)^{-1}\arccos A_j$ $(j = 1, 2)$, we obtain $\sigma_1 = c_1$, $\sigma_2 = c_2$ in g_1, $\sigma_1 = 1 - c_1$, $\sigma_2 = c_2$ in g_2, $\sigma_1 = 1 - c_1$, $\sigma_2 = 1 - c_2$ in g_3, and $\sigma_1 = 1 + c_1$, $\sigma_2 = 1 - c_2$ in g_4.

The non-linear problem. The third-order resonances in the problem of the stability of equilibrium of the body have turned out to be unimportant, since expansion (2.5) contains no third-degree form H_3. It is obvious from the structure of the forms (2.6) and (2.7) that those of the fourth-order resonances (1.30) in which the numbers k_1 and k_2 are odd are also unimportant. Computations have shown that the fourth-order resonances (1.30) in which the numbers k_1 and k_2 have different signs are not realized in the stable domains considered here in the first approximation; when k_1 and k_2 have the same sign, only nine resonances are possible:

$$\begin{gathered}1)\ 4\sigma_1 = 1, \quad 2)\ 2(\sigma_1+\sigma_2) = 1, \quad 3)\ 4\sigma_2 = 1, \quad 4)\ 2(\sigma_1+\sigma_2) = 2, \quad 5)\ 4\sigma_1 = 3 \\ 6)\ 2(\sigma_1+\sigma_2) = 3, \quad 7)\ 4\sigma_2 = 3, \quad 8)\ 2(\sigma_1+\sigma_2) = 4, \quad 9)\ 4\sigma_1 = 5\end{gathered} \tag{2.8}$$

Corresponding to each of the resonance relations (2.8) in the α, β plane there is a curve issuing from a point $(\alpha_m, 0)$ on the $\beta = 0$ axis, where

$$\alpha_1 = 0.0625, \quad \alpha_2 = 0.0858, \quad \alpha_3 = 0.1250, \quad \alpha_4 = 0.3431, \quad \alpha_5 = 0.5625$$

$$\alpha_6 = 0.7721, \quad \alpha_7 = 1.1250, \quad \alpha_8 = 1.3726, \quad \alpha_9 = 1.5625$$

The resonance curves are shown in Fig. 1. There are three resonance curves 1–3 in the domain g_1, one curve 4 in g_2, two curves 5, 6 in g_3, and three curves 7–9 in g_4.

Off the resonance curves (2.8), the Hamiltonian of perturbed motion (2.5) has the normal form (1.44). If

$$D = c_{11}^2 - 4c_{20}c_{02} \neq 0 \tag{2.9}$$

then the equilibrium position in question is stable for most initial conditions (in the sense of Lebesgue measure) [4, 11]. In addition, if the function

$$F(r_1, r_2) = c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 \tag{2.10}$$

is of fixed sign for $r_1 \ge 0$, $r_2 \ge 0$, the equilibrium position is formally stable [4, 9, 12].

For fourth-order resonance, the normalized Hamiltonian has the form (1.46). If

$$|F(k_1, k_2)| > k_1^{k_1/2} k_2^{k_2/2} \sqrt{\alpha_{k_1k_200}^2 + \beta_{k_1k_200}^2} \tag{2.11}$$

the equilibrium position is stable in the third approximation (that is, including terms up to H_4 inclusive in expansion (2.5)). In the case of the opposite inequality, the equilibrium position is unstable in Lyapunov's sense [4].

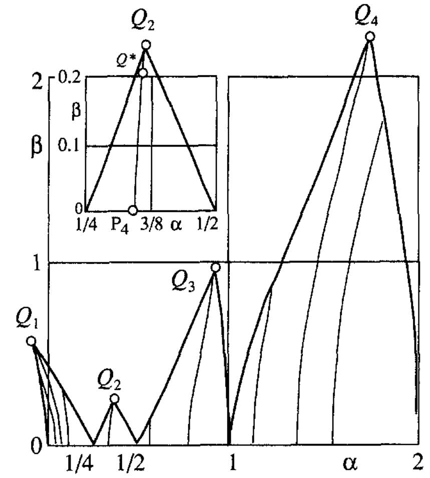

Fig. 1

Computations based on the algorithm of Section 1 have shown that, for values of the parameters α and β off the curves (2.8) in the stability domains in the first approximation, D is negative. For such parameter values, therefore, the relative equilibrium position of the body is stable for most initial conditions, and it is also formally stable.

On all the resonance curves (2.8) except for the curve $2(\sigma_1 + \sigma_2) = 2$ one has inequality (2.11), and on these curves, therefore, one has stability in the third approximation. As to the curve $2(\sigma_1 + \sigma_2) = 2$ itself, it is divided by the point $Q_*(0.3622, 0.2161)$ into stable and unstable segments (see the upper left insert in Fig. 1). On the P_4Q_* segment one has stability in the third approximation, and on the Q_*Q_2 segment the relative equilibrium of the body is unstable in Lyapunov's sense.

3. COMPUTATIONAL FORMULAE

This section presents formulae for computing the coefficients of the normal forms (1.44)–(1.46). For the forms F_k and F_k^* (k = 3, 4), defined by (1.18), (1.19) and (1.24), we introduce the notation

$$F_k = \sum f_{\nu_1\nu_2\mu_1\mu_2} Q_1^{(0)^{\nu_1}} Q_2^{(0)^{\nu_2}} P_1^{(0)^{\mu_1}} P_2^{(0)^{\mu_2}}, \quad F_k^* = \sum f^*_{\nu_1\nu_2\mu_1\mu_2} x_1^{(0)^{\nu_1}} x_2^{(0)^{\nu_2}} y_1^{(0)^{\mu_1}} y_2^{(0)^{\mu_2}} \tag{3.1}$$

For the form F_3^* we have $f^*_{\nu_1\nu_2\mu_1\mu_2} = z_{\nu_1\nu_2\mu_1\mu_2}$, and the following relations hold

$$f^*_{\nu_1\nu_2\mu_1\mu_2} = -\frac{1-i}{4}(a_{\nu_1\nu_2\mu_1\mu_2} + ib_{\nu_1\nu_2\mu_1\mu_2}), \quad f^*_{\mu_1\mu_2\nu_1\nu_2} = -\frac{1-i}{4}(a_{\nu_1\nu_2\mu_1\mu_2} - ib_{\nu_1\nu_2\mu_1\mu_2}) \tag{3.2}$$

$$(a_{3000} = f_{3000} - f_{1020}), \quad (b_{3000} = f_{2010} - f_{0030}), \quad (a_{2100} = f_{2100} - f_{1011} - f_{0120})$$
$$(b_{2100} = f_{2001} + f_{1110} - f_{0021}), \quad (a_{2010} = f_{1020} + 3f_{3000}), \quad (b_{2010} = f_{2010} + 3f_{0030})$$
$$(a_{1110} = 2(f_{2100} + f_{0120})), \quad (b_{1110} = 2(f_{2001} + f_{0021})) \tag{3.3}$$
$$a_{2001} = f_{2100} + f_{1011} - f_{0120}, \quad b_{2001} = f_{0021} + f_{1110} - f_{2001}$$
$$a_{1002} = f_{1200} - f_{1002} + f_{0111}, \quad b_{1002} = f_{0210} - f_{1101} - f_{0012}$$

From this point on, an equality enclosed in parentheses means that, apart from the equality itself, any equality obtained by simultaneous permutation of the first two and last two subscripts also holds. For example, besides the first equality of (3.3), we have the equality $a_{0300} = f_{0300} - f_{0102}$.

The coefficients f^*_{2020}, f^*_{1111} and f^*_{0202} of the form F_4^* are real

$$(f^*_{2020} = -(3f_{4000} + f_{2020} + 3f_{0040})/2), \quad f^*_{1111} = -(f_{2200} + f_{2002} + f_{0220} + f_{0022}) \tag{3.4}$$

The remaining coefficients of the form F_4^* needed for normalization are pairs of complex conjugates

$$f^*_{\nu_1\nu_2\mu_1\mu_2} = \frac{1}{4}(a_{\nu_1\nu_2\mu_1\mu_2} + ib_{\nu_1\nu_2\mu_1\mu_2}),\quad f^*_{\mu_1\mu_2\nu_1\nu_2} = \frac{1}{4}(a_{\nu_1\nu_2\mu_1\mu_2} - ib_{\nu_1\nu_2\mu_1\mu_2})$$

where

$$\begin{aligned}
&(a_{4000} = f_{2020} - f_{4000} - f_{0040}),\quad (b_{4000} = f_{1030} - f_{3010})\\
&(a_{1300} = f_{0211} + f_{1102} - f_{1300} - f_{0013}),\quad (b_{1300} = f_{0112} + f_{1003} - f_{1201} - f_{0310})\\
&a_{2200} = f_{1111} - f_{2200} + f_{2002} - f_{0022} + f_{0220},\quad b_{2200} = f_{1012} - f_{1210} + f_{0121} - f_{2101}
\end{aligned} \tag{3.5}$$

The function W_4 defined by (1.35) may be written as a sum

$$W_4 = \sum w_{\nu_1\nu_2\mu_1\mu_2} \xi_1^{(0)^{\nu_1}} \xi_2^{(0)^{\nu_2}} \eta_1^{(0)^{\mu_1}} \eta_2^{(0)^{\mu_2}} \tag{3.6}$$

and we introduce the following notation

$$c_{\nu_1\nu_2\mu_1\mu_2} = a_{\nu_1\nu_2\mu_1\mu_2} b_{\nu_1\nu_2\mu_1\mu_2},\quad \tilde{c}^{\pm}_{\nu_1\nu_2\mu_1\mu_2} = a^2_{\nu_1\nu_2\mu_1\mu_2} \pm b^2_{\nu_1\nu_2\mu_1\mu_2}$$

$$c^{\pm}_{m_1m_2n_1n_2r_1r_2s_1s_2} = a_{m_1m_2n_1n_2} b_{r_1r_2s_1s_2} \pm a_{r_1r_2s_1s_2} b_{m_1m_2n_1n_2}$$

$$\hat{c}^{\pm}_{m_1m_2n_1n_2r_1r_2s_1s_2} = a_{m_1m_2n_1n_2} a_{r_1r_2s_1s_2} \pm b_{m_1m_2n_1n_2} b_{r_1r_2s_1s_2}$$

Relations (1.35), (1.23), (1.24), (1.31) and (3.1)–(3.5) yield the following expressions for the coefficients of the function (1.36)

$$\begin{aligned}
w_{2020} = {}& f^*_{2020} + \frac{1}{8}(4c_{2010} + 3c^-_{20103000} + c^-_{20012100} + c_{1110}) - \frac{1}{16}\{3\operatorname{ctg}(\pi\sigma_1)\tilde{c}^+_{2010} +\\
&+ \operatorname{ctg}(\pi\sigma_2)\tilde{c}^+_{1110} + 9\operatorname{ctg}(3\pi\sigma_1)\tilde{c}^+_{3000} + \operatorname{ctg}[\pi(2\sigma_1+\sigma_2)]\tilde{c}^+_{2100} - \operatorname{ctg}[\pi(2\sigma_1-\sigma_2)]\tilde{c}^+_{2001}]
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
w_{1111} = {}& f^*_{1111} + \frac{1}{4}(c^-_{10021101} + c^+_{11102001} + c^+_{02011110} + c^+_{11012010} + c^-_{11102100} + c^-_{11011200}) -\\
&- \frac{1}{4}\{\operatorname{ctg}(\pi\sigma_1)\hat{c}^+_{20101101} + \operatorname{ctg}(\pi\sigma_2)\hat{c}^+_{11100201} + \operatorname{ctg}[\pi(2\sigma_1+\sigma_2)]\tilde{c}^+_{2100} +\\
&+ \operatorname{ctg}[\pi(\sigma_1+2\sigma_2)]\tilde{c}^+_{1200} + \operatorname{ctg}[\pi(2\sigma_1-\sigma_2)]\tilde{c}^+_{2001} - \operatorname{ctg}[\pi(\sigma_1-2\sigma_2)]\tilde{c}^+_{1002}\}
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
w_{0202} = {}& f^*_{0202} + \frac{1}{8}(4c_{0201} + 3c^-_{02010300} + c^+_{12001002} + c_{1101}) - \frac{1}{16}\{3\operatorname{ctg}(\pi\sigma_2)\tilde{c}^+_{0201} +\\
&+ \operatorname{ctg}(\pi\sigma_1)\tilde{c}^+_{1101} + 9\operatorname{ctg}(3\pi\sigma_2)\tilde{c}^+_{0300} + \operatorname{ctg}[\pi(\sigma_1+2\sigma_2)]\tilde{c}^+_{1200} + \operatorname{ctg}[\pi(\sigma_1-2\sigma_2)]\tilde{c}^+_{1002}\}
\end{aligned} \tag{3.9}$$

For the real and imaginary parts of the resonance coefficients (1.39) we have the following expressions

$$\begin{aligned}
\mu_{4000} = {}& \frac{1}{4}a_{4000} + \frac{1}{16}(9c_{3000} + c_{2100} - c_{2010} - c_{2001}) +\\
&+ \frac{1}{16}\{3\operatorname{ctg}(\pi\sigma_1)\hat{c}^-_{30002010} + \operatorname{ctg}[\pi(2\sigma_1-\sigma_2)]\hat{c}^-_{21002001}\}
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\nu_{4000} = {}& -\frac{1}{4}b_{4000} + \frac{1}{32}(9\tilde{c}^-_{3000} + \tilde{c}^-_{2100} - \tilde{c}^-_{2010} - \tilde{c}^-_{2001}) -\\
&- \frac{1}{16}\{3\operatorname{ctg}(\pi\sigma_1)c^+_{20103000} + \operatorname{ctg}[\pi(2\sigma_1-\sigma_2)]c^+_{20012100}\}
\end{aligned} \tag{3.11}$$

$$\begin{aligned} \mu_{0400} &= \frac{1}{4}a_{0400} + \frac{1}{16}(9c_{0300} + c_{1200} + c_{1002} - c_{0201}) + \\ &+ \frac{1}{16}\{3\operatorname{ctg}(\pi\sigma_2)\hat{c}^{-}_{03000201} - \operatorname{ctg}[\pi(\sigma_1 - 2\sigma_2)]\hat{c}^{+}_{10021200}\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \nu_{0400} &= -\frac{1}{4}b_{0400} + \frac{1}{32}(9\tilde{c}^{-}_{0300} + \tilde{c}^{-}_{1200} - \tilde{c}^{-}_{1002} - \tilde{c}^{-}_{0201}) - \\ &- \frac{1}{16}\{3\operatorname{ctg}(\pi\sigma_2)c^{+}_{02010300} - \operatorname{ctg}[\pi(\sigma_1 - 2\sigma_2)]c^{-}_{10021200}\} \end{aligned} \tag{3.13}$$

$$\begin{aligned} \mu_{2200} &= \frac{1}{4}a_{2200} + \frac{1}{16}[4(c_{2100} + c_{1200}) + 3(c^{+}_{12003000} + c^{+}_{03002100}) - c_{1110} - c_{1101} + \\ &+ c^{-}_{20101002} - c^{+}_{20010201}] + \frac{1}{16}[\operatorname{ctg}(\pi\sigma_1)(2\hat{c}^{-}_{11011200} + \hat{c}^{-}_{12002010}) + \\ &+ \operatorname{ctg}(\pi\sigma_2)(2\hat{c}^{-}_{11102100} + \hat{c}^{-}_{21000201}) - 3\operatorname{ctg}(3\pi\sigma_1)\hat{c}^{+}_{30001002} - 3\operatorname{ctg}(3\pi\sigma_2)\hat{c}^{-}_{03002001}] \end{aligned} \tag{3.14}$$

$$\begin{aligned} \nu_{2200} &= -\frac{1}{4}b_{2200} + \frac{1}{32}[6(\hat{c}^{-}_{12003000} + \hat{c}^{-}_{03002100}) + 4(\tilde{c}^{-}_{2100} + \tilde{c}^{-}_{1200}) - \\ &- 2(\hat{c}^{+}_{20101002} + \hat{c}^{-}_{02012001}) - \tilde{c}^{-}_{1101} - \tilde{c}^{-}_{1110}] - \frac{1}{16}[\operatorname{ctg}(\pi\sigma_1)(2c^{+}_{11011200} + c^{+}_{12002010}) + \\ &+ \operatorname{ctg}(\pi\sigma_2)(2c^{+}_{11102100} + c^{+}_{21000201}) - 3\operatorname{ctg}(3\pi\sigma_1)c^{-}_{10023000} - 3\operatorname{ctg}(3\pi\sigma_2)c^{+}_{03002001}] \end{aligned} \tag{3.15}$$

$$\begin{aligned} \mu_{1300} &= \frac{1}{4}a_{1300} + \frac{1}{16}(6c^{+}_{12000300} + 2c^{+}_{21001200} - c^{+}_{02011101} + c^{-}_{11101002}) + \\ &+ \frac{1}{16}[3\operatorname{ctg}(\pi\sigma_1)\hat{c}^{+}_{03001101} + \operatorname{ctg}(\pi\sigma_2)(2\hat{c}^{-}_{12000201} + \hat{c}^{-}_{12001110}) + 2\operatorname{ctg}(5\pi\sigma_2)\hat{c}^{+}_{10022100}] \end{aligned} \tag{3.16}$$

$$\begin{aligned} \nu_{1300} &= -\frac{1}{4}b_{1300} + \frac{1}{16}(6\hat{c}^{-}_{03001200} + 2\hat{c}^{-}_{12002100} - \hat{c}^{+}_{10021110} - \hat{c}^{-}_{02011101}) - \\ &- \frac{1}{16}[3\operatorname{ctg}(\pi\sigma_1)c^{+}_{03001101} + \operatorname{ctg}(\pi\sigma_2)(2c^{+}_{12000201} + c^{+}_{12001110}) - 2\operatorname{ctg}(5\pi\sigma_2)c^{-}_{21001002}] \end{aligned} \tag{3.17}$$

$$\begin{aligned} \mu_{3100} &= \frac{1}{4}a_{3100} + \frac{1}{16}(6c^{+}_{21003000} + 2c^{+}_{12002100} - c^{+}_{20101110} - c^{+}_{11012001}) + \\ &+ \frac{1}{16}[3\operatorname{ctg}(\pi\sigma_2)\hat{c}^{-}_{30001110} + \operatorname{ctg}(\pi\sigma_1)(2\hat{c}^{-}_{21002010} + \hat{c}^{-}_{21001101}) + 2\operatorname{ctg}(5\pi\sigma_1)\hat{c}^{-}_{12002001}] \end{aligned} \tag{3.18}$$

$$\begin{aligned} \nu_{3100} &= -\frac{1}{4}b_{3100} + \frac{1}{16}(6\hat{c}^{-}_{30002100} + 2\hat{c}^{-}_{12002100} - \hat{c}^{-}_{20101110} - \hat{c}^{-}_{11012001}) - \\ &- \frac{1}{16}[3\operatorname{ctg}(\pi\sigma_2)c^{+}_{30001110} + \operatorname{ctg}(\pi\sigma_1)(2c^{+}_{21002010} + c^{+}_{21001101}) + 2\operatorname{ctg}(5\pi\sigma_1)c^{+}_{20011200}] \end{aligned} \tag{3.19}$$

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